Can a lambda-model have a recursively enumerable theory ?

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Can a λ-model have an r.e. theory?

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Types and Computations: A Colloquium in Honor of
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Outline

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Syntax of the untyped λ-calculus

Terms of untyped λ-calculus

λ-terms: $M, N ::= x \mid MN \mid \lambda x.M$

Examples:

$\mathbf{I} \equiv \lambda x.x,\ \mathbf{T} \equiv \lambda xy.x,\ \mathbf{F} \equiv \lambda xy.y,\ \Omega \equiv (\lambda x.xx)(\lambda x.xx).$

β-conversion

$(\lambda x.M)N =_\beta M[x := N].$

λ-theory

Any congruence on Λ containing β-conversion.

Examples of λ-theories

$\lambda\beta$ = the smallest λ-theory,

$\lambda\beta\eta$ = the smallest extensional λ-theory.

The lattice of λ-theories

- λ-theories form a complete lattice $(\lambda\mathcal{T}, \subseteq)$.

There is a continuum of λ-theories

$card(\lambda\mathcal{T}) = 2^{\aleph_0}$: the set of r.e. λ-theories is *dense* in $\lambda\mathcal{T}$.
$\mathcal{T}$ is r.e. if $\{(M, N) : M =_{\mathcal{T}} N\}$ is r.e.

General aim

Understanding the structure of $\lambda\mathcal{T}$.

Remark

Few λ-theories come from syntactical considerations $(\lambda\beta, \lambda\beta\eta, BT, \mathcal{H}^*)$, most come as the equational theory of a model:

$$Eq(\mathcal{M}) = \{(M, N) : \mathcal{M} \models M = N\}$$

AIM: Investigate this conjecture

Conjecture

No (proper = non-syntactical) λ-calculus model belonging to the "main semantics":

- Continuous semantics (Scott),
  simplest subclass: **Graph models** (never extensional),
- Stable semantics (Berry and Girard),
- Strongly stable semantics (Bucciarelli and Ehrhard),

can have an r.e. equational (order) theory.

Recall

All these models are *partially ordered* with a $\perp$ element.

Can a proper model have an r.e. theory?

Every (uniform) class of p.o. models:

- (Kerth 1995, Gouy, Bastonero 1996) represents $2^{\aleph_0}$ λ-theories,
- (Salibra 2001) omits $2^{\aleph_0}$ λ-theories.

Remark

Most models *do not* have an r.e. theory:

- there are only countably many r.e. theories,
- $\mathcal{T}$ sensible $\Rightarrow \mathcal{T}$ non r.e.,
- the few known theories of models ($BT, \mathcal{H}^*$ & co.) are non-r.e.

Is this question interesting?

A longstanding open problem...

Problem (Honsell $\simeq$1984)

$\exists$? a proper model $\mathcal{M}$ such that *Eq*$(\mathcal{M}) = \lambda\beta, \lambda\beta\eta$ (r.e.)?

Theorem (Selinger 1995)

If *Eq*$(\mathcal{M}) = \lambda\beta, \lambda\beta\eta$ then any p.o. on $(\mathcal{M}, \bullet)$ is trivial on (the interpretations of) closed λ-terms.

Known results

- **NO!** for graph models (corollary).
- (DiG-H-P,1995) $\exists$ proper model $\mathcal{M}$ such that *Eq*$(\mathcal{M}) = \lambda\beta\eta$ in a weakly continuous semantics and not strictly proper!!!
- (BBB 98-00) $\exists$ strictly proper model $\mathcal{M}$ of System F in the Scott-continuous semantics such that *Eq*$(\mathcal{M}) = \lambda\beta\eta^F$ Typed calculus!!!

Outline

Methodology

- We concentrate on the following two classes of p.o. models:
  1. $\mathcal{E}ff$ = "Effective" λ-models (based on effective domains)
     Reason 1: Effective models (in our sense) are omni-present in the literature.
     Reason 2: The BBB model of System F, is effective in our sense.
  2. $\mathcal{G}$ = Graph models (Based on $(\mathcal{P}(D), \subseteq)$)
     Reason: simplest class, deeply studied, generic (in some sense).
- We also look at order theories:
  $Ord(\mathcal{M}) = \{(M, N) : \mathcal{M} \models M \sqsubseteq N\}$.
  Note that: $Ord(\mathcal{M})$ r.e. $\Rightarrow Eq(\mathcal{M})$ r.e.
- (Visser 1980) If $U, V \subseteq \Lambda^o$ are β-closed, co-r.e. and non-empty, then $U \cap V \neq \emptyset$.

Outline

Effective λ-models

Effective domains

An effective domain is a Scott-domain $\mathcal{D}$ equipped with an adequate notion of computable elements:
$\mathcal{D}^{r.e.}$ = computable elements of $\mathcal{D}$;
$\mathcal{D}^{dec}$ = decidable elements of $\mathcal{D}$; [BMS07]

(Scott-continuous) Effective λ-models

A λ-model $\mathcal{M} = (\mathcal{D}, A, \lambda)$ with $\lambda, A : [\mathcal{D} \to \mathcal{D}] \lhd \mathcal{D}$ is *weakly effective* if $\mathcal{D}$ is an effective Scott domain, and A, λ are computable.
$\mathcal{M}$ is *effective* modulo a further technical condition on A, λ.

All the models introduced individually in the literature can be proved effective in our sense...

More details

Effective domains

An effective domain is a Scott-domain $\mathcal{D} = (D, \sqsubseteq_{\mathcal{D}}, d)$ where $d : \mathbb{N} \to K(D)$ is surjective such that:

- the relation "$d_n \sqcup d_m$ exists" is decidable in (m, n),
- the relation $d_k = d_n \sqcup d_m$ is decidable in (n, m, k).

Computable and decidable elements

$x \in \mathcal{D}^{r.e.}$ iff $\{n : d_n \sqsubseteq x\}$ is r.e.
$x \in \mathcal{D}^{dec}$ iff $\{n : d_n \sqsubseteq x\}$ is decidable.

Computable functions

$f \in [\mathcal{D} \to \mathcal{D}']^{r.e.}$ iff the relation $d'_m \sqsubseteq f(d_n)$ is r.e. in (n, m).

Properties

$\mathcal{M}$ weakly effective

(i) $|M| \in \mathcal{D}^{r.e.}$ for all $M \in \Lambda^o$,

(ii) if $u \in \mathcal{D}^{dec}$ then $u^- = \{N \in \Lambda^o : |N| \sqsubseteq u\}$ is co-r.e.

Proof Sketch: Weak effectivity implies that the interpretation function is computable. Then (i) and (ii) follow.

$\mathcal{M}$ effective

If $M, N \in \Lambda^o$ are β-normal, then:

(i) $|M| \in \mathcal{D}^{dec}$, hence:

(ii) $M^- = \{N \in \Lambda^o : |N| \sqsubseteq |M|\}$ is co-r.e., thus:

(iii) $M^- \cap N^- \neq \emptyset$ by [Visser, 1980].

Can an effective λ-model have an r.e. theory?

$\mathcal{M}$ effective

1. $Ord(\mathcal{M})$ is non r.e.
2. $Eq(\mathcal{M}) \neq \lambda\beta, \lambda\beta\eta$,
3. If $\perp^- \neq \emptyset$ then $Eq(\mathcal{M})$ is non r.e.,
4. If $\mathcal{M}$ is stable or strongly stable then $Eq(\mathcal{M})$ is non r.e.

Remark 1

$T^- \neq F^-$, where $T \equiv \lambda xy.x, F \equiv \lambda xy.y$.

Remark 2

If $\mathcal{M}$ is effective, then:
(i) $\perp^-, T^-, F^-$ are co-r.e. β-closed subsets of Λ^o.
(ii) $T^- \cap F^- \neq \emptyset$.

Proof sketches

If $\mathcal{M}$ is an effective λ-model...

Theorem 1.

$Ord(\mathcal{M})$ is non r.e.

Proof. If $Ord(\mathcal{M})$ is r.e., then M^- is r.e. β-closed and non-empty. Hence, Rem. 2 implies $F^- = T^- = \Lambda^o$. Contrad. Rem. 1!

Theorem 2.

$Eq(\mathcal{M}) \neq \lambda\beta, \lambda\beta\eta$.

Proof. Follows from Rem 2(ii) and Selinger's result, since $N \in T^- \cap F^-$ implies $\mathcal{M} \models N \subsetneq T$.

Proof sketches

If $\mathcal{M}$ is an effective λ-model...

Theorem 3.

If $\perp^- \neq \emptyset$ then $Eq(\mathcal{M})$ is non r.e.,

Proof. $\perp^- \subsetneq \Lambda^o$ is a β-closed, co-r.e. set by Rem. 2(i). Moreover $\perp^-$ is just 1 equivalence class! Hence $Eq(\mathcal{M})$ cannot be r.e.

Theorem 4.

$\mathcal{M}$ stable or strongly stable implies $Eq(\mathcal{M})$ non r.e.

Proof. Follows from Thm. 3 since in the stable and strongly stable semantics $T^- \cap F^- = \perp^- \neq \emptyset$ (nearly true).

Outline

Graph models (Scott-continuous semantics)

- **Kernel of the definition.** A *graph model* is generated by a "web" (D, i), where:
  - D is an infinite set,
  - $i : \mathcal{P}_f(D) \times D \to D$ is a *total* injection.

  The underlying reflexive cpo is $(\mathcal{P}(D), \subseteq)$.
- **Furthermore:** there is a "*free completion*" process for generating a web (and hence a graph model) from a partial pair (A, j). Ex: Engeler, P_ω,...

Löwenheim Skolem for graph models

For all graph models $\mathcal{G}$ there exists $\mathcal{G}'$ with countable web such that $Eq(\mathcal{G}) = Eq(\mathcal{G}')$ (and $Ord(\mathcal{G}) = Ord(\mathcal{G}')$).
$\Rightarrow$ Wlog we can suppose D countable or even $D = \mathbb{N}$.

Effective graph models

- $\mathcal{G}$ graph model with web $(\mathbb{N}, i)$.
- *Effective domain:* $\mathcal{D} = (\mathcal{P}(\mathbb{N}), \subseteq)$,
  computable elements: r.e. subsets,
  decidable elements: decidable subsets.
- $\mathcal{M} = ((P(D), \subseteq), A, \lambda)$,
  - $\lambda, A : [P(D) \to P(D)] \lhd P(D)$ are defined by:
  - $\lambda(f) \equiv \{i(d, \alpha) : d \in P_f(D) \text{ and } \alpha \in f(d)\}$,
  - $A(u)(v) \equiv \{\alpha \in D : (\exists d \subseteq_f v) i(d, \alpha) \in u\}$.

Effective graph models

Proposition

(i) If i is computable and *dom*(i) is decidable, then $\mathcal{G}$ is weakly effective.
(ii) If furthermore *range*(i) is decidable, then $\mathcal{G}$ is effective.

Examples: All free completions of finite partial pairs.

Similar for all classes of webbed models

K-models (Krivine) $\supset$ Scott's and Park's models are effective.
F-ilter models (Coppo-Dezani-Honsell-Longo-Barendregt),
G-models (Girard's reflexive coherent spaces),
H-models (Ehrhard's reflexive hypercoherences),

Can a graph theory be r.e.?

Theorem 1

There is a minimal equational/order graph theory, which happens to be the theory of a (non unique) effective model $\mathcal{G}$.

Theorem 2

If $\mathcal{G}$ is a graph model then $Ord(\mathcal{G})$ is not r.e.

Theorem 3

If $\mathcal{G}$ is an effective graph model generated by a partial pair finite modulo its group of automorphisms, then $Eq(\mathcal{G})$ is not r.e.

Corollary

No graph model generated by a finite partial pair can have an r.e. theory.

Conclusions

- The weaker conjecture for order theories is:
  1. Proved for all graph models,
  2. Proved for all effective models.
- Concerning effective models the conjecture is:
  1. Proved for the *stable* and *strongly stable* semantics.
  2. Open for the continous semantics,
  3. Even for the restricted case of graph models.
  4. But our results cover all the models individually introduced in the literature.
- Concerning graph models the conjecture is:
  Proved for large families of effective graph models.

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